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TRANSLATION GENERALIZED QUADRANGLES IN EVEN CHARACTERISTIC

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In this paper, we first introduce new objects called "translation generalized ovals" and "translation generalized ovoids", and make a thorough study of these objects. We then obtain numerous new characterizations of the $T_2(\mathcal{O})$ of Tits and the classical generalized quadrangle $\mathcal{Q}(4,q)$ in even characteristic, including the complete classification of 2-transitive generalized ovals for the even case. Next, we prove a new strong characterization theorem for the $T_3(\mathcal{O})$ of Tits. As a corollary, we obtain a purely geometric proof of a theorem of Johnson on semifield flocks.

1. Introduction

A (finite) *generalized quadrangle (GQ) of order* (s,t) is an incidence structure $\mathcal{S}=(P,B,I)$ in which P and B are disjoint (nonempty) sets of objects called 'points' and 'lines' respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(i) Each point is incident with $t+1$ lines $(t\geq 1)$ and two distinct points are incident with at most one line.
(ii) Each line is incident with $s+1$ points $(s\geq 1)$ and two distinct lines are incident with at most one point.
(iii) If p is a point and L is a line not incident with p, then there is a unique point-line pair (q,M) such that $pIMIqIL$.

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If $s = t$, then $\mathcal{S}$ is also said to be of *order* s. If $s > 1$ and $t > 1$, then we call the GQ *thick*.

Generalized quadrangles were introduced by Jacques Tits [34] in his celebrated work on triality as a subclass of a larger class of incidence structures, namely the *generalized polygons*, in order to provide a geometric interpretation of the Chevalley groups of rank 2. The main results up to 1983 on finite generalized quadrangles are contained in the monograph *Finite Generalized Quadrangles* [15] (denoted "FGQ" in the course of this paper) by S. E. Payne and J. A. Thas.

Suppose $\mathcal{S}$ is a finite generalized quadrangle of order (s,t), $s \neq 1 \neq t$, and consider a point x. Then $\mathcal{S} = \mathcal{S}^{(x)}$ is said to be a *translation generalized quadrangle (TGQ)* with *base-point* x if there is an abelian group G of collineations (defined as usual) of $\mathcal{S}$ fixing x linewise and acting regularly on the set of points which are not collinear with x. The group G is called the *base-group* or *translation group* of the TGQ.

Let L be a line of the GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$. A *symmetry about* L is an automorphism of $\mathcal{S}$ fixing each line that meets L. The line L is an *axis of symmetry* if the group of all symmetries about L has size s. In that case, it is easy to see that L is regular (cf. Section 2) – see Chapter 8 (Section 8.1) of FGQ. A point each line through which is an axis of symmetry is a *translation point*. Dually, we define *symmetries about points* and *centers of symmetry*.

Theorem 1.1 (FGQ, 8.3.1). *The GQ $\mathcal{S}$ of order (s,t), $s \neq 1 \neq t$, is a TGQ with base-point x if and only if x is a translation point. In that case, the base-group of the TGQ is uniquely defined as the group generated by all symmetries about the lines through x.*

It is the main goal of the present paper to obtain characterizations and classifications of translation generalized quadrangles in even characteristic.

The paper is organized as follows. In § 4, we introduce a theory for *translation generalized ovals/ovoids* in $\mathbf{PG}(2n+m-1,q)$, thus generalizing the concept of translation oval in $\mathbf{PG}(2,q)$ (q even). We study these objects in some detail, show that they can only occur in the generalized oval case, and characterize them geometrically. In § 6 we obtain several new characterizations of the $T_2(\mathcal{O})$ of Tits of order q, q even. In § 9 we show that a generalized ovoid $\mathcal{O}$ of $\mathbf{PG}(4n-1,q)$, q even, which is good at some element $\mathbf{PG}(n-1,q)$, is regular if and only if the $q^{2n}+q^n$ pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$ are regular. In § 10, we give a geometrical proof of a well-known theorem of N. L. Johnson, as a corollary of the main result of § 9.

2. Finite Generalized Quadrangles: Basic Theory

2.1. Combinatorial theory

Let $\mathcal{S}=(P,B,I)$ be a (finite) thick generalized quadrangle of order (s,t). Then $|P|=(s+1)(st+1)$ and $|B|=(t+1)(st+1)$ [15]; also, $s\leq t^2$ and, dually, $t\leq s^2$.

There is a point-line duality for GQ's of order (s,t) for which in any definition or theorem the words "point" and "line", and the parameters s and t are interchanged. The dual of the GQ $\mathcal{S}$ will be denoted by $\mathcal{S}^D$. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

If $\mathcal{S}$ is a GQ of order $(s,1)$, then $\mathcal{S}$ is also called a *grid*.

Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p\sim q$ and say that p and q are *collinear*, provided that there is some line L so that $pILIq$ (so $p\not\sim q$ means that p and q are not collinear). Dually, for $L,M\in B$, we write $L\sim M$ or $L\not\sim M$ according as L and M are *concurrent* or *non-concurrent* respectively. If $p\neq q\sim p$, the line incident with both points is denoted by pq, and if $L\sim M\neq L$, the point which is incident with both lines is sometimes denoted by $L\cap M$.

For $p\in P$, we put $p^\perp=\{q\in P \parallel q\sim p\}$, and we note that $p\in p^\perp$. For a pair of distinct points $\{p,q\}$, the *trace* of $\{p,q\}$ is defined by $p^\perp\cap q^\perp$, and we denote this set by $\{p,q\}^\perp$. Then $|\{p,q\}^\perp|=s+1$ or $t+1$, according as $p\sim q$ or $p\not\sim q$. More generally, if $A\subset P$, $A^\perp$ is defined by $A^\perp=\bigcap\{p^\perp \parallel p\in A\}$. For $p\neq q$, the *span* of the pair $\{p,q\}$ is $\{p,q\}^{\perp\perp}=\{r\in P \parallel r\in s^\perp$ for all $s\in\{p,q\}^\perp\}$. Then $|\{p,q\}^{\perp\perp}|=s+1$ or $|\{p,q\}^{\perp\perp}|\leq t+1$ according as $p\sim q$ or $p\not\sim q$. If $p\sim q$, $p\neq q$, or if $p\not\sim q$ and $|\{p,q\}^{\perp\perp}|=t+1$, we say that the pair $\{p,q\}$ is *regular*. The point p is *regular* provided $\{p,q\}$ is regular for every $q\in P\setminus\{p\}$. One easily proves that either $s=1$ or $t\leq s$ if $\mathcal{S}$ has a regular pair of non-collinear points, see FGQ. Regularity for lines is defined dually.

A *triad* of points is a triple of pairwise non-collinear points. Given a triad T, a *center* of T is just an element of $T^\perp$. If a triad has at least one center, then we say that it is *centric*. A *panel* (p,L,q) of $\mathcal{S}=(P,B,I)$ is an element of $P\times B\times P$ for which $pILIq\neq p$. A *subquadrangle*, or also *subGQ*, $\mathcal{S}'=(P',B',I')$ of a GQ $\mathcal{S}=(P,B,I)$ of order (s,t), $s,t>1$, is a GQ for which $P'\subseteq P$, $B'\subseteq B$, and where I' is the restriction of I to $(P'\times B')\cup(B'\times P')$.

2.2. Classical generalized quadrangles arising from quadrics

Consider a nonsingular quadric of Witt index 2, that is, of projective index 1, in $\mathbf{PG}(4,q)$ and $\mathbf{PG}(5,q)$, respectively. The points and lines of the quadric form a generalized quadrangle which is denoted by $\mathcal{Q}(4,q)$ and $\mathcal{Q}(5,q)$, respectively, and has order (q,q) and (q,q^2), respectively.

The points of $\mathbf{PG}(3,q)$ together with the totally isotropic lines with respect to a symplectic polarity, form a GQ $W(q)$ of order q.

The following result will be used without further notice.

Theorem 2.1 (FGQ, 3.2.1). *For q any prime power, $\mathcal{Q}(4,q)\cong W(q)^D$.*

3. Translation Generalized Quadrangles, Generalized Ovoids and Generalized Ovals

3.1. Generalized ovoids and generalized ovals

Suppose $H=\mathbf{PG}(2n+m-1,q)$ is the finite projective $(2n+m-1)$-space over $\mathbf{GF}(q)$. Now define a set $\mathcal{O}=\mathcal{O}(n,m,q)$ of subspaces as follows: $\mathcal{O}$ is a set of q^m+1 $(n-1)$-dimensional subspaces of H, denoted $\mathbf{PG}^{(i)}(n-1,q)$, $i\in\{0,1,\dots,q^m\}$, so that

(i) every three subspaces generate a $\mathbf{PG}(3n-1,q)$,
(ii) for every $i\in\{0,1,\dots,q^m\}$ there is a subspace $\mathbf{PG}^{(i)}(n+m-1,q)$ of H of dimension $n+m-1$, which contains $\mathbf{PG}^{(i)}(n-1,q)$ and which is disjoint from each $\mathbf{PG}^{(j)}(n-1,q)$ if $j\neq i$.

If $\mathcal{O}$ satisfies these conditions for $n=m$, then $\mathcal{O}$ is called a *pseudo-oval* or a *generalized oval* or an $[n-1]$-*oval* of $\mathbf{PG}(3n-1,q)$. A generalized oval of $\mathbf{PG}(2,q)$ is just an oval of $\mathbf{PG}(2,q)$. For $n\neq m$, $\mathcal{O}(n,m,q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of $\mathbf{PG}(2n+m-1,q)$. A [0]-ovoid of $\mathbf{PG}(3,q)$ is just an ovoid of $\mathbf{PG}(3,q)$. The space $\mathbf{PG}^{(i)}(n+m-1,q)$ is called the *tangent space* (or just the *tangent*) *to* $\mathcal{O}(n,m,q)$ *at* $\mathbf{PG}^{(i)}(n-1,q)$.

Generalized ovals were introduced by J. A. Thas in [20], and generalized ovoids by S. E. Payne and J. A. Thas in FGQ, Chapter 8. In [21,15], S. E. Payne and J. A. Thas prove that from an $\mathcal{O}=\mathcal{O}(n,m,q)$ there arises a GQ $T(n,m,q)=T(\mathcal{O})$. This GQ is a TGQ of order (q^n,q^m) with base-point (∞), and is constructed as follows. Embed H in a $\mathbf{PG}(2n+m,q)=H'$.

- Points are of three types:
  1. a symbol (∞);
  2. the subspaces $\mathbf{PG}(n+m,q)$ of H' which intersect H in a $\mathbf{PG}^{(i)}(n+m-1,q)$;
  3. the points of $H'\setminus H$.

- LINES are of two types:
  1. the elements of $\mathcal{O}(n,m,q)$;
  2. the subspaces $\mathbf{PG}(n,q)$ of $\mathbf{PG}(2n+m,q)$ which intersect H in an element of $\mathcal{O}(n,m,q)$.
- INCIDENCE is defined as follows: the point (∞) is incident with all the lines of type (1) and with no other lines; a point of type (2) is incident with the unique line of type (1) contained in it and with all the lines of type (2) which it contains (as subspaces), and finally, a point of type (3) is incident with the lines of type (2) that contain it.

Conversely, any TGQ is isomorphic to the $T(n,m,q)$ associated with an $\mathcal{O}(n,m,q)$ in $\mathbf{PG}(2n+m-1,q)$, and so the study of translation generalized quadrangles is essentially equivalent to the study of generalized ovals and ovoids; see Chapter 8 in FGQ.

The following theorem is due to S. E. Payne and J. A. Thas; see Chapter 8 of FGQ.

Theorem 3.1. *For any TGQ of order (s,t), $s \neq 1 \neq t$, we either have $s=t$ or $t=s^{\frac{k+1}{k}}$, with k odd. If s is even, then $t=s$ or $t=s^2$.*

Each TGQ $\mathcal{S}$ of order (s,t) with translation point (∞), where $s \neq 1 \neq t$, has a *kernel* $\mathbb{K}$, where $\mathbb{K}$ is a field whose multiplicative group is isomorphic to the group of all collineations of $\mathcal{S}$ fixing linewise the point (∞), and any given point not collinear with (∞); see FGQ. We have $|\mathbb{K}| \leq s$, see FGQ. The field $\mathbf{GF}(q)$ is a subfield of $\mathbb{K}$ if and only if $\mathcal{S}$ is of type $T(n,m,q)$. The TGQ $\mathcal{S}$ is isomorphic either to a $T_3(\mathcal{O})$ of Tits with $\mathcal{O}$ an ovoid of $\mathbf{PG}(3,s)$, or to a $T_2(\mathcal{O})$ of Tits with $\mathcal{O}$ an oval of $\mathbf{PG}(2,s)$, if and only if $|\mathbb{K}|=s$.

In the extension $\mathbf{PG}(2n+m-1,q^n)$ of $\mathbf{PG}(2n+m-1,q)$, with $m \in \{n,2n\}$, we consider n $(\frac{m}{n}+1)$-dimensional spaces of $\mathbf{PG}^{(i)}(\frac{m}{n}+1,q^n) = \pi_i$, with $i=1,2,\dots,n$, which are conjugate with respect to the extension $\mathbf{GF}(q^n)$ of $\mathbf{GF}(q)$ and which span $\mathbf{PG}(2n+m-1,q^n)$. In π_1, we now consider an oval $\mathcal{O}_1$ for $m=n$ and an ovoid $\mathcal{O}_1$ for $m=2n$. Let $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \dots, x_{q^m}^{(1)}\}$. Further, let $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$, with $i=0,1,\dots,q^m$, be conjugate with respect to the extension $\mathbf{GF}(q^n)$ of $\mathbf{GF}(q)$. The points $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$ define an $(n-1)$-dimensional space $\mathbf{PG}^{(i)}(n-1,q)$ over $\mathbf{GF}(q)$. Then $\{\mathbf{PG}^{(0)}(n-1,q), \mathbf{PG}^{(1)}(n-1,q), \dots, \mathbf{PG}^{(q^m)}(n-1,q)\}$ is a generalized oval of $\mathbf{PG}(3n-1,q)$ for $m=n$, and a generalized ovoid of $\mathbf{PG}(4n-1,q)$ for $m=2n$. Here, we speak of a *regular pseudo-oval*, respectively a *regular pseudo-ovoid*. For $m=n$, any known $[n-1]$-oval is regular, for $m=2n$ and q even any known $[n-1]$-ovoid is regular, but for $m=2n$ and q odd there are $[n-1]$-ovoids which are not regular; see J. A. Thas [24]. A regular pseudo-oval for which $\mathcal{O}_1$ is a conic is called either a *pseudo-conic* or a *classical pseudo-oval*, and a regular pseudo-ovoid for which $\mathcal{O}_1$ is an elliptic quadric is called a *pseudo-quadric* or a classical *pseudo-ovoid* or a *classical egg*. If $\mathcal{O}$ is a pseudo-conic, then $T(\mathcal{O}) \cong T_2(\mathcal{O}_1) \cong \mathcal{Q}(4,q^n)$; if $\mathcal{O}$ is a pseudo-quadric, then $T(\mathcal{O}) \cong T_3(\mathcal{O}_1) \cong \mathcal{Q}(5,q^n)$.

3.2. Translation duals

If $n \neq m$, then by 8.7.2 of FGQ the q^m+1 tangent spaces to $\mathcal{O}(n,m,q)$ form an egg $\mathcal{O}^*(n,m,q)$ in the dual space of $\mathbf{PG}(2n+m-1,q)$. So in addition to $T(n,m,q)$ there arises a TGQ $T(\mathcal{O}^*)$, also denoted $T^*(n,m,q)$, or $T^*(\mathcal{O})$. The TGQ $T^*(\mathcal{O})$ is called the *translation dual* of the TGQ $T(\mathcal{O})$.

Now let $n=m$, with q odd. Then similar observations can be made, see e.g. [15, p. 182]. So also in this case there arises a *translation dual* of the TGQ $T(\mathcal{O})$. However, the only known such TGQ is the classical GQ $\mathcal{Q}(4,s)$, which is isomorphic to its translation dual.

3.3. Good TGQ's; good eggs

A TGQ $T(\mathcal{O})$ with $t=s^2$, $s=q^n$, respectively an egg $\mathcal{O}(n,2n,q)$, is called *good* at an element $\pi \in \mathcal{O}$ if for every two distinct elements π' and π'' of $\mathcal{O} \setminus \{\pi\}$ the $(3n-1)$-space $\pi\pi'\pi''$ contains exactly q^n+1 elements of $\mathcal{O}$; in such a case π is called a *good element* of $\mathcal{O}$. From [15, 8.7.2 (v)] then easily follows that $\pi\pi'\pi''$ is skew to the other elements of $\mathcal{O}$.

4. Translation Generalized Ovoids and Translation Generalized Ovals

Let θ be an involution in the projective general linear automorphism group $\mathbf{PGL}(k+1,q)$ of the projective space $\mathbf{PG}(k,q)$, with q even. Then the set of all fixed points of θ is a subspace $\mathbf{PG}(r,q)$ of $\mathbf{PG}(k,q)$, called the *axis* of θ. The fixed hyperplanes for θ are all hyperplanes containing some subspace $\mathbf{PG}(k-r-1,q)$ of $\mathbf{PG}(k,q)$, which is called the *center* of θ. Also, $\mathbf{PG}(k-r-1,q) \subseteq \mathbf{PG}(r,q)$ and so $k \leq 2r+1$. If π_1 and π_2 are mutually skew $(k-r+1)$-dimensional subspaces of $\mathbf{PG}(k,q)$ which are skew to a given $\mathbf{PG}(r,q) \subseteq \mathbf{PG}(k,q)$, then there is exactly one involution θ of $\mathbf{PG}(k,q)$ with axis $\mathbf{PG}(r,q)$ mapping π_1 onto π_2; the center of θ is the subspace $\langle \pi_1, \pi_2 \rangle \cap \mathbf{PG}(r,q)$. Further, all involutions of $\mathbf{PG}(k,q)$ with given axis $\mathbf{PG}(r,q)$ form an elementary abelian group.

For more details on involutions of finite projective spaces we refer to Chapter 16 of [18]. Notice that in [18] the term "fundamental space" is used instead of "axis", and the term "fundamental axis" instead of "center".

Let $\mathcal{O}$ be a generalized ovoid in $\mathbf{PG}(2n+m-1,q)$, $n \neq m$, respectively a generalized oval in $\mathbf{PG}(3n-1,q)$, with q even. Then $\mathcal{O}$ is a *translation generalized ovoid*, respectively a *translation generalized oval*, with *axis* the tangent space $\mathbf{PG}^{(i)}(n+m-1,q)$ at $\mathbf{PG}^{(i)}(n-1,q) \in \mathcal{O}$, if there is a group of involutions of $\mathbf{PGL}(2n+m,q)$ with axis $\mathbf{PG}^{(i)}(n+m-1,q)$, fixing $\mathcal{O}$ and acting regularly on $\mathcal{O} \setminus \{\mathbf{PG}^{(i)}(n-1,q)\}$.

If $n=m=1$, then a translation generalized oval is just called a *translation oval*; if $2n=m=2$, it is called a *translation ovoid*. All translation ovals of $\mathbf{PG}(2,q)$, $q=2^h$, were determined by S. E. Payne in [11]; up to recoordinatization, they are always of the form

$$\{(1,t,t^{2^i}) \parallel t \in \mathbf{GF}(q)\} \cup \{(0,0,1)\},$$

where i is fixed in $\{1,2,\ldots,h-1\}$ and $(h,i)=1$.

Suppose $\mathcal{S}=(P,B,I)$ is a TGQ of order q^n with translation point (∞). Let $x \not\sim (\infty)$ be a regular point. Then each point of $P \setminus (\infty)^{\perp}$ is regular. Whence each point of $\mathcal{S}$ is regular, and $\mathcal{S} \cong W(q^n)$ by 5.2.1 of FGQ.

Theorem 4.1 (Structure of Translation Generalized Ovoids and Ovals).

(i) *A translation generalized ovoid in* $\mathbf{PG}(2n+m-1,q)$*, q even, does not exist for $n \neq m$.*

(ii) *Let $\mathcal{O}$ be a non-classical generalized oval in* $\mathbf{PG}(3n-1,q)$*, with q even. Then the following are equivalent.*

- (a) *$\mathcal{S}=T(\mathcal{O})$ contains a regular point x different from the translation point (which is necessarily collinear with (∞));*
- (b) *the point-line dual $\mathcal{S}^D$ of $\mathcal{S}$ is a TGQ (with as base-point the line $x(\infty)$ of $\mathcal{S}$);*
- (c) *$\mathcal{O}$ is a translation generalized oval with axis the tangent to $\mathcal{O}$ at its element $x(\infty)$.*

Proof. Let $\mathcal{O}$ be a non-classical generalized oval in $\mathbf{PG}(3n-1,q)$. Suppose x is a regular point different from (∞). Then by the remark preceding the theorem, x is collinear with (∞). Hence each point $yIx(\infty)$, $y \neq (\infty)$, is regular. By 1.5.2(iv) of FGQ, also the point (∞) is regular. The translations of $T(\mathcal{O})$ fixing the point x, induce elations, with as axis the set of points of $x(\infty)$ of the projective plane Π_x defined by the regular point x of $\mathcal{S}$. Hence Π_x is a translation plane having as translation line the set of points on $x(\infty)$. Then by J. A. Thas [27], S^D is a TGQ where the base-point is the line $x(\infty)$ of $\mathcal{S}$. So $\mathcal{S}$ contains a regular point x different from the translation point if and only $x \sim (\infty)$ and $\mathcal{S}^D$ is a TGQ with as base-point the line $x(\infty)$ of $\mathcal{S}$.

Suppose that $\mathcal{S}^D$ is a TGQ with as base-point the line $x(\infty)$ of $\mathcal{S}$. Then the translation group of $\mathcal{S}^D$ has a subgroup of size q^n acting regularly on the lines, distinct from $x(\infty)$, incident with (∞). Hence $\mathcal{O}$ is a translation generalized oval with as axis the tangent to $\mathcal{O}$ at its element $x(\infty)$.

Finally, suppose that $\mathcal{O}$ is a translation generalized oval with axis the tangent τ to $\mathcal{O}$ at π, where π is the line $x(\infty)$. Let $G(\mathcal{O})$ be the group of involutions with axis τ which acts regularly on $\mathcal{O} \setminus \{\pi\}$. Embed $\mathbf{PG}(3n-1,q) = \langle \mathcal{O} \rangle$ in $\mathbf{PG}(3n,q)$. Let η be an arbitrary $2n$-dimensional subspace of $\mathbf{PG}(3n,q)$ which is not contained in $\mathbf{PG}(3n-1,q)$, and which contains τ. Then there is a group G' of involutions of $\mathbf{PG}(3n,q)$ with axis η which induces $G(\mathcal{O})$ in $\mathbf{PG}(3n-1,q)$; the corresponding automorphism group of $T(\mathcal{O})$ will also be denoted by G'. The axis η is a point of $T(\mathcal{O})$ and G' fixes $\eta^{\perp}$ pointwise. As $|G'| = q^n$, the point η is a center of symmetry, and so is a regular point (see Section 8.1 of FGQ). Part (ii) of the theorem is proved.

Now suppose that $\mathcal{O}$ is a translation generalized ovoid in $\mathbf{PG}(2n+m-1,q)$, $n \neq m$, with axis Π (Π being the tangent to $\mathcal{O}$ at the element π). Then by the final part of the proof of Part (ii), each $(n+m)$-dimensional subspace of $\mathbf{PG}(2n+m,q) \supset \mathbf{PG}(2n+m-1,q)$ which is not contained in $\mathbf{PG}(2n+m-1,q)$ and which contains Π, is a regular point of $T(\mathcal{O})$. Hence $s \geq t$, so $n \geq m$, contradiction. The theorem is proved. ∎

Remark 4.2 (On the Notion of 'Translation Generalized Oval/ Ovoid').

(a) Let $\mathcal{O}$ be a generalized ovoid, respectively generalized oval, in $\mathbf{PG}(2n+m-1,q)$, and let $\pi \in \mathcal{O}$. Suppose Π is the tangent to $\mathcal{O}$ at π. Suppose there is a subgroup G of $\mathbf{P\Gamma L}(2n+m,q)$ that fixes Π pointwise and that acts regularly on $\mathcal{O} \setminus \{\pi\}$. We will call this Property (*) for the moment. Then by an argument similar to the one in the final part of the proof of Theorem 4.1, each $(n+m)$-dimensional subspace of $\mathbf{PG}(2n+m,q) \supset \mathbf{PG}(2n+m-1,q)$ which is not contained in $\mathbf{PG}(2n+m-1,q)$ and which contains Π, is a regular point of $T(\mathcal{O})$. So $n=m$. By 1.5.2(i) of FGQ, it follows that q is even. By 1.5.2(iv) of FGQ also the point (∞) is regular. Let $LI(\infty)$ be the line of regular points of $T(\mathcal{O})$ that corresponds to the space Π of $\mathbf{PG}(3n-1,q)$. Then in the same way as in the first part of the proof of Theorem 4.1, the dual $T(\mathcal{O})^D$ of $T(\mathcal{O})$ is a TGQ with as base-point the line L. It follows that G is an elementary abelian group, so that each nontrivial element of G is an involution.

Hence a generalized oval, respectively ovoid, satisfies Property (*) if and only if it is a translation generalized oval, respectively ovoid. This is the reason why we only defined translation generalized ovals and ovoids in even characteristic.

(b) Although we have shown that there are no translation generalized ovoids, we still prefer to define these objects without restriction on the parameters n and m. A reason is that in the infinite case one then also can use this definition. Also, it is not excluded that in some proof translation generalized ovoids will turn up (then leading to a contradiction).

Corollary 4.3 **(Chapter 12 of FGQ; See also K. Thas [33]).** *Let $\mathcal{O}$ be a non-classical oval in $\mathbf{PG}(2,q)$, q even. Then $\mathcal{S}=T_2(\mathcal{O})$ contains a regular point x different from the translation point if and only if $x\sim(\infty)$ and $\mathcal{S}^D$ is a $T_2(\mathcal{O})$ with base-point the point corresponding to $x(\infty)$, if and only if $\mathcal{O}$ is a translation oval with axis the tangent to $\mathcal{O}$ at its element $x(\infty)$.* ∎

Remark 4.4. J. Tits [35] defines a *translation ovoid ("ovoïde à translation") $\mathcal{O}$* of $\mathbf{PG}(n,\mathbb{K})$, $n=2,3$ and $\mathbb{K}$ a (not necessarily finite) field, in a slightly different way; here $\mathcal{O}$ is a set of points such that for each point $x\in\mathcal{O}$ the union of the tangent lines to $\mathcal{O}$ at x is a hyperplane of $\mathbf{PG}(n,\mathbb{K})$. He demands that *for each tangent T to $\mathcal{O}$*, there is a group of elations with axis T, fixing $\mathcal{O}$ globally, and acting regularly on the points of $\mathcal{O}$ not incident with T. Amongst many other results, he then proves that $\mathbf{PG}(n,\mathbb{K})$, $n=2,3$, contains a translation ovoid if and only if the characteristic of $\mathbb{K}$ is 2 and there is a subfield $\mathbb{L}$ of $\mathbb{K}$ for which $(n-1)[\mathbb{K}:\mathbb{L}]\leq[\mathbb{L}:\mathbb{L}^2]$. Suppose that $\mathbb{K}$ is finite, i.e., that it is isomorphic to $\mathbf{GF}(2^r)$ for some r. If $n=3$, then clearly no such subfield exists, so $\mathbf{PG}(3,2^r)$ does not contain translation ovoids in the sense of Tits. If $n=2$, then the only possibility is that $\mathbb{L}=\mathbb{K}=\mathbf{GF}(2^r)$, and in this case the translation ovoid is a conic of $\mathbf{PG}(2,\mathbb{K})$.

Let $\mathcal{S}=T(\mathcal{O})$ be a TGQ of order s, with $s=q^n$ even. Let $\mathcal{O}\subset\mathbf{PG}(3n–1,q)$, and write $\mathcal{O}=\{\pi,\pi_1,\ldots,\pi_{q^n}\}$. Suppose $\mathfrak{N}$ is the *nucleus* of $\mathcal{O}$, that is, the common $(n-1)$-dimensional space of all tangents to $\mathcal{O}$ (see Section 8.7 of FGQ, Section 4.9 of [20]). Consider the $(2n-1)$-space $\mathit{\Pi}=\langle\pi,\mathfrak{N}\rangle$ (that is, the tangent space at π to $\mathcal{O}$). For each $i\in\{1,2,\ldots,q^n\}$, the set

$$\{\pi,\mathfrak{N}\}\cup\{\langle\pi_i,\pi_j\rangle\cap\mathit{\Pi}\parallel i\neq j\}$$

is an $(n-1)$-spread of $\mathit{\Pi}$, denoted $\mathbf{S}_i$.

Note that if $\mathcal{O}$ is a translation generalized oval with axis $\langle\pi,\mathfrak{N}\rangle$, then all the $\mathbf{S}_i$'s coincide. It is also important to note that if all the $\mathbf{S}_i$'s coincide, we have that if γ is an element of that spread distinct from π and $\mathfrak{N}$, and if $j \in \{1,2,\ldots,q^n\}$, then $\langle\gamma,\pi_j\rangle$ contains *precisely one other* element π_r of $\mathcal{O}$ (and hence is disjoint from π_k if $k \neq j,r$). Suppose all the $\mathbf{S}_i$'s coincide. We say that $\mathcal{O}$ is *projective at* $\langle\pi,\mathfrak{N}\rangle$ if the following property holds:

> *Let γ be an element of $\mathbf{S} = \mathbf{S}_i$ (for all i), where $\pi \neq \gamma \neq \mathfrak{N}$, and let $j \neq k$ be in $\{1,2,\ldots,q^n\}$ such that $\langle\gamma,\pi_j\rangle \neq \langle\gamma,\pi_k\rangle$. By the above, there are elements $\pi_{j'}$ and $\pi_{k'}$ so that $\pi_{j'} \subseteq \langle\gamma,\pi_j\rangle$ and $\pi_{k'} \subseteq \langle\gamma,\pi_k\rangle$. Then $\langle\pi_j,\pi_k\rangle \cap \langle\pi_{k'},\pi_{j'}\rangle$ is an element of $\mathbf{S}$.*

Remark 4.5. Each translation oval $\mathcal{O}$ of $\mathbf{PG}(2,s)$ with axis L, s even, has the (in that case trivial) property that the $\mathbf{S}_i$'s coincide, and also is projective at L. It should be noticed that for a generalized oval the fact that the $\mathbf{S}_i$'s coincide does not imply that $\mathcal{O}$ is projective; this follows from Theorem 4.6 below, and the existence of ovals in $\mathbf{PG}(2,q)$, q even, which are not translation ovals.

Suppose all the $\mathbf{S}_i$'s coincide, and that the generalized oval $\mathcal{O}$ is projective at $\langle\pi,\mathfrak{N}\rangle$. Interpret $\mathcal{O}$ over $\mathbf{GF}(2)$, that is, if $q = 2^m$, then we consider in $\mathbf{PG}(3nm-1,2)$ the corresponding generalized oval, which will also be denoted by $\mathcal{O}$ (and we keep using the same notation as before). Then it is clear that the $\mathbf{S}_i$'s still coincide. Let π_j and π_k be arbitrary distinct elements of $\mathcal{O}\backslash\{\pi\}$, and suppose that $\langle\pi_j,\pi_k\rangle \cap \mathit{\Pi} = \mathit{\Pi}_{jk}$. Let θ_{jk} be the involution with axis $\mathit{\Pi}$, center $\mathit{\Pi}_{jk}$, which maps π_j onto π_k. Then θ_{jk} fixes each $(2nm-1)$-subspace of $\mathbf{PG}(3nm-1,2)$ containing $\mathit{\Pi}_{jk}$, fixes $\mathcal{O}$ globally, and sends π_j onto π_k. If then G is the group consisting of all the θ_{jk}'s, $j \neq k$, $j,k \in 1,2,\ldots,2^{nm}$, together with the identity, then it is clear that G is a group of involutions with axis $\mathit{\Pi}$, acting regularly on $\mathcal{O} \setminus \{\pi\}$. Whence $\mathcal{O}$ is a translation generalized oval.

So we have the following theorem.

Theorem 4.6. *Let $\mathcal{O}$ be a generalized oval in $\mathbf{PG}(3n-1,q)$, q even, and use the above notation. Then $\mathcal{O}$ is a translation generalized oval with axis $\langle\pi,\mathfrak{N}\rangle$ if and only if all the $\mathbf{S}_i$'s coincide and $\mathcal{O}$ is projective at $\langle\pi,\mathfrak{N}\rangle$.* ∎

Note that for $n = 1$, this is the well-known geometrical characterization of translation ovals in $\mathbf{PG}(2,q)$ (q even).

5. Note on the Definition of Translation Generalized Oval/Ovoid

In this section, we make the following useful observation.

Theorem 5.1. *A generalized oval $\mathcal{O}$ of $\mathbf{PG}(3n-1,q)$, q even, is a translation generalized oval with axis τ, where τ is the tangent space to $\mathcal{O}$ at $\pi \in \mathcal{O}$, if and only if there is a subgroup of involutions of $\mathbf{PGL}(3n,q)$ which fixes π and acts regularly on the remaining elements of $\mathcal{O}$.*

Proof. Let G be a group of involutions as stated. Let θ be a nontrivial element of G. Suppose η is the kernel of $\mathcal{O}$. Let Π be an arbitrary n-dimensional subspace of $\mathbf{PG}(3n-1,q)$ containing π but not contained in τ, and suppose χ is a $(2n-1)$-dimensional subspace of $\mathbf{PG}(3n-1,q)$ containing η and meeting τ in η. First of all, one notes that $\Pi^\theta \neq \Pi$. For, suppose that this is not the case. Let π' be the unique element of $\mathcal{O}\setminus\{\pi\}$ intersecting Π. Then ${\Pi'}^\theta = \Pi'$, clearly a contradiction.

For each $p \in \chi$, put $p^\alpha = \langle p,\pi\rangle^\theta \cap \chi$. Then α is a linear involutory automorphism of χ. As α does not fix any point of $\chi\setminus\eta$, it follows that α must fix each point of η (as the dimension of η is $n-1$). But θ and α have the same action on η, so that θ must fix η pointwise. By the same argument, θ also fixes π pointwise, so that τ is fixed pointwise by θ.

Now suppose $\zeta \in \mathcal{O}\setminus\{\pi\}$. Then $\langle\zeta,\zeta^\theta\rangle \cap \langle\pi,\eta\rangle$ belongs to the center, so that the center has dimension at least $n-1$. So the dimension of the axis is at most $2n-1$, from which follows that τ *is* the axis of θ. The theorem readily follows. ∎

From the proof of Theorem 5.1 we have the following useful observation:

Observation 5.2. *Let $\mathcal{O}$ be a generalized oval $\mathcal{O}$ of $\mathbf{PG}(3n-1,q)$, q even, and let τ be the tangent space to $\mathcal{O}$ at the element $\pi \in \mathcal{O}$. Suppose that there is an involution in $\mathbf{PGL}(3n,q)$ which stabilizes $\mathcal{O}$, which fixes π and does not fix any of the remaining elements of $\mathcal{O}$. Then θ is an involution with axis τ.* ∎

Remark 5.3. It should be remarked that Theorem 5.1 is not true for generalized ovoids. For, consider the GQ $\mathcal{Q}(5,q^n)$, q even, and suppose p is a point of $\mathcal{Q}(5,q^n)$. Then p is a translation point, and $Aut(\mathcal{Q}(5,q^n))_p$ contains a subgroup H which acts naturally as $\mathbf{PSL}(2,q^{2n})$ on the set of lines incident with p. In particular, for any line $L\mathrm{I}p$, the stabilizer of L in H contains a normal elementary abelian 2-subgroup which acts regularly on the lines incident with p and distinct from L. So, if $\mathcal{O}$ is the generalized ovoid in $\mathbf{PG}(4n-1,q)$ corresponding to the TGQ $\mathcal{Q}(5,q^n)^{(p)}$, and τ is the tangent space to $\mathcal{O}$ at $\pi \in \mathcal{O}$, where π corresponds to L, then there is a subgroup G of involutions of $\mathbf{PGL}(4n,q)$ which fixes π and acts regularly on the remaining elements of $\mathcal{O}$. But G can not fix τ pointwise by Theorem 4.1.

6. Characterizations of the $T_2(\mathcal{O})$ of Tits

Not many characterizations of the $T_2(\mathcal{O})$ of Tits, $\mathcal{O}$ an oval in $\mathbf{PG}(2,s)$ with s even, are known (see, for example, Chapters 8 and 12 of FGQ for some results). It is the goal of this section to provide new characterizations of these objects so that, in combination with Theorem 9.1 below, we can obtain far-reaching classification results for TGQ's of order (s,s^2), s even.

We now have

Theorem 6.1. *Let $\mathcal{S}=T(\mathcal{O})$ be a TGQ of order s, $s=q^n$ even, with base-point (∞). Put $\mathcal{O}=\{\pi,\pi_1,\dots,\pi_{q^n}\}$, and define $\mathbf{S}_i$ as above, $i=1,2,\dots,q^n$. If for at least two distinct i and j in $\{1,2,\dots,q^n\}$, the spreads $\mathbf{S}_i$ and $\mathbf{S}_j$ coincide, and $\mathbf{S}_i=\mathbf{S}_j$ is regular (that is, the projective plane defined by the regular line π_i, respectively π_j, is Desarguesian), then $\mathcal{S}\cong T_2(\mathcal{O}')$ for some oval $\mathcal{O}'$ of $\mathbf{PG}(2,s)$, and conversely.*

Proof. If $\mathcal{S}\cong T_2(\mathcal{O}')$ for some oval $\mathcal{O}'$ of $\mathbf{PG}(2,s)$, then $\mathbf{S}_i=\mathbf{S}_j$ for all $i\neq j$ and clearly for any k the spread $\mathbf{S}_k$ is regular.

Conversely, suppose that $\mathcal{S}$ is as in the theorem. We use the notation preceding the theorem. First of all, as $\mathbf{S}_i=\mathbf{S}_j$ is Desarguesian, there are precisely n lines over $\mathbf{GF}(q^n)$ which intersect the extensions of the spread elements of $\mathbf{S}_i$ in precisely one point. Fix such a line L^*. Consider $\langle\pi_i,\pi_j\rangle\cap\mathit{\Pi}=\mathit{\Pi}_{ij}$, and let M^* be the unique line over $\mathbf{GF}(q^n)$ through the point of L^* in the extension of $\mathit{\Pi}_{ij}$ to $\mathbf{GF}(q^n)$, which intersects the extensions of π_i and π_j. Let $\mathit{\Gamma}^*$ be the Desarguesian projective plane over $\mathbf{GF}(q^n)$ which is generated by L^* and M^*. Now let π_k be an element of $\mathcal{O}$, $i\neq k\neq j$. Then $\langle\pi_i,\pi_k\rangle\cap\mathit{\Pi}=\mathit{\Pi}_{ik}$ and $\langle\pi_j,\pi_k\rangle\cap\mathit{\Pi}=\mathit{\Pi}_{jk}$ are elements of $\mathbf{S}_i=\mathbf{S}_j$, distinct from $\pi,\mathfrak{N}$ and $\mathit{\Pi}_{ij}$, where $\mathfrak{N}$ is the nucleus of $\mathcal{O}$. Let U^* be the line of $\mathit{\Gamma}^*$ which is incident with the point of L^* in the extension of $\mathit{\Pi}_{ik}$ to $\mathbf{GF}(q^n)$, and with the point of M^* in the extension of π_i to $\mathbf{GF}(q^n)$. Let V^* be the line of $\mathit{\Gamma}^*$ which is incident with the point of L^* in the extension of $\mathit{\Pi}_{jk}$ to $\mathbf{GF}(q^n)$, and with the point of M^* in the extension of π_j to $\mathbf{GF}(q^n)$. Then over $\mathbf{GF}(q^n)$, $U^*\cap V^*$ is a point of the extension of $\langle\pi_i,\pi_k\rangle\cap\langle\pi_j,\pi_k\rangle=\pi_k$ to $\mathbf{GF}(q^n)$. We have shown that over $\mathbf{GF}(q^n)$, each element of $\mathcal{O}$ meets $\mathit{\Gamma}^*$. Hence over $\mathbf{GF}(q^n)$, there are n Desarguesian planes which all meet each element of $\mathcal{O}$, implying that $\mathcal{O}$ can be embedded in a regular $(n-1)$-spread of $\mathbf{PG}(3n-1,q)$. The theorem follows. ∎

Theorem 6.2. *Let $\mathcal{S}^{(\infty)}=T(\mathcal{O})$ be a TGQ of order s with base-point (∞). Suppose there is some line $L\mathrm{I}(\infty)$ for which the associated projective plane $\mathit{\Pi}_L$ is Desarguesian. Suppose that $x\neq(\infty)$ is a regular point, $x\not\mathrm{I}L$. Then $\mathcal{O}$ is a regular translation generalized oval, i.e. $\mathcal{S}\cong T_2(\mathcal{O}')$ of Tits for some translation oval $\mathcal{O}'$ of $\mathbf{PG}(2,s)$, and conversely.*

Proof. First suppose that x is not collinear with (∞). Then since $\mathcal{S}^{(\infty)}$ is a TGQ, each point not collinear with (∞) is regular. It follows that each point is regular, and then $\mathcal{S}^{(\infty)} \cong W(s)$ by 5.2.1 of FGQ. Also, s is even by 1.5.1(ii) of FGQ. The theorem follows.

Now suppose $x \sim (\infty)$. By 1.5.2(i) of FGQ s is even. Then Theorem 4.1 implies that $\mathcal{O}$ is a translation generalized oval with axis the tangent to $\mathcal{O}$ at $x(\infty)$. The result now follows from Theorems 4.6 and 6.1.

The converse is obvious. ∎

The following theorem is very general.

Theorem 6.3. *Let $\mathcal{S} = \mathcal{S}^{(\infty)} = T(\mathcal{O})$ be a TGQ of order s, s even, with base-point (∞). Suppose there is a line $L\mathrm{I}(\infty)$ for which the associated projective plane Π_L is Desarguesian. Suppose also that there is a line $M \neq L$ which is incident with (∞), and an involution θ of $Aut(\mathcal{S})_{(\infty)}$ which fixes M pointwise and which does not fix any of the lines incident with (∞) and different from M. Then $\mathcal{S} \cong T_2(\mathcal{O}')$ of Tits for some oval $\mathcal{O}'$ of $\mathbf{PG}(2,s)$.*

Proof. Let π, respectively π', be the element of $\mathcal{O}$ corresponding to L, respectively M, and let Π' be the tangent to $\mathcal{O}$ at π'. Suppose θ is as in the theorem. Then by Observation 5.2 and the proof of Theorem 5.1, θ fixes Π' pointwise and does not fix any element of $\mathcal{O} \setminus \{\pi'\}$. So π and π^θ induce the same spread $\mathbf{S}$ in Π'. Moreover, since Π_L is Desarguesian, we have that $\mathbf{S}$ is regular. Now apply Theorem 6.1. ∎

Corollary 6.4. *Let $\mathcal{S} = \mathcal{S}^{(\infty)} = T(\mathcal{O})$ be a TGQ of order s, s even, with base-point (∞). Suppose there is a line $L\mathrm{I}(\infty)$ for which the associated projective plane Π_L is Desarguesian. Suppose that there is a nontrivial symmetry about some point $x \not\mathrm{I} L$. Then $\mathcal{S} \cong T_2(\mathcal{O}')$ of Tits for some oval $\mathcal{O}'$ of $\mathbf{PG}(2,s)$.*

Proof. If x is not collinear with (∞) then one observes that each point of $\mathcal{S}$ is a translation point, so that each point is regular. The corollary then follows from 5.2.1 and 3.2.2 of FGQ.

If $x \sim (\infty)$, the corollary follows from Theorem 6.3. ∎

7. A Characterization of Translation Generalized Ovals

Let $\mathcal{O} = \{\pi, \pi_1, \ldots, \pi_{q^n}\}$ be a generalized oval in $\mathbf{PG}(3n-1, q)$, with $q = 2^h$ even, and let τ be the tangent to $\mathcal{O}$ at π. Now we define a point-line incidence structure $\mathbf{A}(\mathcal{O})$ as follows:

- POINTS are the elements of $\mathcal{O} \setminus \{\pi\}$;

- LINES are the pairs $\{\pi_i, \pi_j\}$ with $i, j \in \{1, 2, \ldots, q^n\}$ and $i \neq j$;
- INCIDENCE is containment.

Hence $\mathbf{A}(\mathcal{O})$ is the complete graph with vertex set $\mathcal{O} \setminus \{\pi\}$. Further, two lines $\{\pi_i, \pi_j\}$ and $\{\pi_k, \pi_l\}$ are called *parallel* if $\langle \pi_i, \pi_j \rangle \cap \tau = \langle \pi_k, \pi_l \rangle \cap \tau$.

Theorem 7.1. *The incidence structure $\mathbf{A}(\mathcal{O})$ provided with parallelism is isomorphic to the hn-dimensional affine space $\mathbf{AG}(hn, 2)$ over $\mathbf{GF}(2)$ if and only if $\mathcal{O}$ is a translation generalized oval with axis τ.*

Proof. Assume that $\mathbf{A}(\mathcal{O})$ provided with parallelism is isomorphic to $\mathbf{AG}(hn, 2)$. If $\{\pi_i, \pi_j\}$ is a line of $\mathbf{A}(\mathcal{O})$ and $\pi_k \notin \{\pi_i, \pi_j\}$, then there is exactly one line of $\mathbf{A}(\mathcal{O})$ containing π_k and parallel to $\{\pi_i, \pi_j\}$. Also, if the distinct lines $\{\pi_i, \pi_j\}$ and $\{\pi_k, \pi_l\}$ are parallel, then also $\{\pi_i, \pi_k\}$ and $\{\pi_j, \pi_l\}$ are parallel. Hence, by Theorem 4.6, $\mathcal{O}$ is a translation generalized oval with axis τ.

Conversely, assume that $\mathcal{O}$ is a translation generalized oval with axis τ. Then, by Theorem 4.6, the q^n $(n-1)$-spreads of τ defined by the respective elements of $\mathcal{O} \setminus \{\pi\}$ coincide, and further $\mathcal{O}$ is projective at π. Now by [9] the point-line incidence structure $\mathbf{A}(\mathcal{O})$ provided with parallelism is isomorphic to $\mathbf{AG}(hn, 2)$. ∎

Remark 7.2. Let $\mathbf{PG}(hn, 2)$ be the projective completion of the affine space $\mathbf{A}(\mathcal{O})$. Then the points at infinity of $\mathbf{A}(\mathcal{O})$, that is, the points of $\mathbf{PG}(hn, 2)$ not in $\mathbf{A}(\mathcal{O})$, can be identified with the elements of $\mathbf{S} \setminus \{\pi, \mathfrak{N}\}$, where $\mathbf{S}$ is the common $(n-1)$-spread of τ defined by the elements of $\mathcal{O} \setminus \{\pi\}$ and $\mathfrak{N}$ is the kernel of $\mathcal{O}$. Any line of $\mathbf{PG}(hn-1, 2) = \mathbf{PG}(hn, 2) \setminus \mathbf{A}(\mathcal{O})$ is of the form $\{\alpha, \beta, \gamma\}$, with $\alpha = \langle \pi_i, \pi_j \rangle \cap \langle \pi_k, \pi_l \rangle$, $\beta = \langle \pi_i, \pi_k \rangle \cap \langle \pi_j, \pi_l \rangle$, $\gamma = \langle \pi_i, \pi_l \rangle \cap \langle \pi_j, \pi_k \rangle$ and $\pi_i, \pi_j, \pi_k, \pi_l$ distinct elements of $\mathcal{O} \setminus \{\pi\}$.

8. Classification of 2-Transitive Generalized Ovals in Even Characteristic

We call a generalized oval $\mathcal{O}$ in $\mathbf{PG}(3n-1, q)$ 2-*transitive* if there is a subgroup of $\mathbf{P\Gamma L}(3n, q)$ which stabilizes $\mathcal{O}$ and acts 2-transitively on its elements. Note that by L. Bader, G. Lunardon and I. Pinneri [1], or J. A. Thas and K. Thas [30], this is equivalent to asking that the automorphism group of $T(\mathcal{O})$, which we may demand to fix the translation point (∞) (as otherwise $T(\mathcal{O}) \cong \mathcal{Q}(4, q^n)$), acts 2-transitively on the lines incident with (∞).

We now classify the 2-transitive generalized ovals in even characteristic. We will look at the equivalent problem for the corresponding TGQ. From the proof will follow which groups arise.

Theorem 8.1. *Let $\mathcal{S}=\mathcal{S}^{(\infty)}=T(\mathcal{O})$ be the TGQ of order (q^n,q^n), q even, which arises from a 2-transitive generalized oval $\mathcal{O}$ in $\mathbf{PG}(3n-1,q)$. Then $\mathcal{S}\cong\mathcal{Q}(4,q^n)$, and so $\mathcal{O}$ is classical.*

Proof. Since q is even, the point (∞) of $T(\mathcal{O})$ is regular. The plane $\pi^D_{(\infty)}$, which is the dual of $\pi_{(\infty)}$, is a translation plane of order q^n, where the parallel classes correspond to the lines of $\mathcal{S}$ incident with (∞). We suppose that $Aut(\mathcal{S})$ fixes (∞) for reasons of conveniency (otherwise $\mathcal{S}\cong\mathcal{Q}(4,q^n)$). Then $Aut(\mathcal{S})$ acts 2-transitively on the parallel classes of $\pi^D_{(\infty)}$. We have the following possibilities by R.-H. Schulz [17] and T. Czerwinsky [5], see also H. Lüneburg [10] (Chapter VI):

(a) $\pi^D_{(\infty)}$ is Desarguesian, and $Aut(\mathcal{S})$ has a subgroup K which induces $\mathbf{PSL}(2,q^n)$ on the line at infinity of $\pi^D_{(\infty)}$;

(b) $\pi^D_{(\infty)}$ is a Lüneburg plane (so that n is even and h is odd, where $q=2^h$), and $Aut(\mathcal{S})$ has a subgroup K which induces the Suzuki group $\mathbf{Sz}(\sqrt{q^n})$ on the line at infinity of $\pi^D_{(\infty)}$.

We take K in such a way that the translation group G of $\mathcal{S}$ is contained in K.

Let α be an involution of K which fixes the element $\pi\in\mathcal{O}$ but not all elements of $\mathcal{O}$; then π is the only element of $\mathcal{O}$ which is fixed by α. We regard α as being an element of $\mathbf{P\Gamma L}(3n,q)$. Since h is odd, α is a linear involution. By Observation 5.2, α is an involution with axis τ, the latter being the tangent space of $\mathcal{O}$ at π. As $\mathbf{PSL}(2,q^n)$ and $\mathbf{Sz}(\sqrt{q^n})$ both are generated by their involutions, we have proved that the nucleus η of $\mathcal{O}$ is fixed pointwise by K (regarded as a subgroup of $\mathbf{PGL}(3n,q)$).

Let z' be a point of $\mathcal{S}$ not collinear with (∞). Then since $\mathcal{S}$ is a TGQ, it is easy to see that $K_{z'}=K'$ still induces $\mathbf{PSL}(2,q^n)$, respectively $\mathbf{Sz}(\sqrt{q^n})$, on the lines incident with (∞). Let $LI(\infty)$ be arbitrary, and consider a Sylow 2-subgroup S_2 of K'_L. As S_2 – seen as a subgroup of $\mathbf{PGL}(3n,q)$ – fixes η pointwise, S_2 fixes each line of S incident with the point z'' on L which is collinear with z' (and S_2 acts sharply transitively on the set of lines incident with (∞) and different from L). If we interpret S_2 as an automorphism group of π_L, it thus follows that S_2 also must fix L pointwise (since in a finite projective plane any central collineation is also axial, and conversely).

The group $G_{z''}$ has order q^{2n} and fixes L pointwise, so together with S_2, it generates a group H such that, if N is the group of symmetries about L, H/N acts sharply transitively on the set of spans $\{U,V\}^{\perp\perp}$ of nonconcurrent lines $U,V\in L^\perp$, and fixes L pointwise. So H/N is a translation group of π^D_L of order q^{2n}, and hence it is elementary abelian. In particular $S_2\cong S_2N/N$

also is. So $\mathbf{PGL}(3n,q)_{\mathcal{O}}$ has a subgroup of involutions that fixes π and acts sharply transitively on $\mathcal{O}\setminus\{\pi\}$.

By Theorem 5.1, we conclude that $\mathcal{O}$ is a translation generalized oval with axis π. So any point incident with L is regular. As L was arbitrary, it follows that each point of $(\infty)^{\perp}$ is regular, so that each point of $\mathcal{S}$ is regular (cf. 1.3.6(iv) of FGQ). It follows that $\mathcal{S}\cong\mathcal{Q}(4,q^n)$ by 5.2.1 and 3.2.2 of FGQ. ∎

9. Good Eggs in $\mathbf{PG}(4n-1,q)$, q Even, for which the Pseudo-Ovals Containing the Good Element are Regular

In this section we will show that a generalized ovoid $\mathcal{O}$ of $\mathbf{PG}(4n-1,q)$, q even, which is good at some element $\mathbf{PG}(n-1,q)$, is regular if the $q^{2n}+q^n$ pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$ are regular.

Theorem 9.1. *Let $\mathcal{O}$ be an egg in $\mathbf{PG}(4n-1,q)$, q even, which is good at $\mathbf{PG}(n-1,q)$. If the $q^{2n}+q^n$ pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$ are regular, then $\mathcal{O}$ is regular.*

Proof. Let $\mathcal{O}=\{\mathbf{PG}(n-1,q),\mathbf{PG}^{(1)}(n-1,q),\dots,\mathbf{PG}^{(q^{2n})}(n-1,q)\}$ be an egg in $\mathbf{PG}(4n-1,q)$, q even, which is good at $\mathbf{PG}(n-1,q)$. Suppose that the $q^{2n}+q^n$ pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$ are regular. Clearly we may assume that $n>1$. Next, assume that $q^n=4$, so $q=n=2$. As $q^n=4$, the $q^{2n}+q^n$ regular pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$ are pseudo-conics. Now by M. R. Brown and M. Lavrauw [2] (see also J. A. Thas [28]) the egg is classical, hence regular. From now on we assume that $q^n>4$.

Let $\mathbf{PG}(3n-1,q)$ be a subspace of $\mathbf{PG}(4n-1,q)$ which is skew to $\mathbf{PG}(n-1,q)$ and let $\langle\mathbf{PG}(n-1,q),\mathbf{PG}^{(i)}(n-1,q)\rangle\cap\mathbf{PG}(3n-1,q)=\mathbf{PG}^{(i)'}(n-1,q)$. Let τ be the tangent space to $\mathcal{O}$ at $\mathbf{PG}(n-1,q)$, and let τ_i be the tangent space to $\mathcal{O}$ at $\mathbf{PG}^{(i)}(n-1,q)$, with $i=1,2,\dots,q^{2n}$. If $\tau'=\tau\cap\mathbf{PG}(3n-1,q)$, then the intersections of τ' with the tangent spaces at $\mathbf{PG}(n-1,q)$ to the pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$, form an $(n-1)$-spread S^* of τ'. Then $S^*\cup\{\mathbf{PG}^{(1)'}(n-1,q),\dots,\mathbf{PG}^{(q^{2n})'}(n-1,q)\}$ is a regular $(n-1)$-spread S of $\mathbf{PG}(3n-1,q)$. If we extend the elements of S to $\mathbf{GF}(q^n)$, then these extensions have a point in common with each of n planes $\pi_1,\pi_2,\dots,\pi_n$ over $\mathbf{GF}(q^n)$.

Let $\mathbf{PG}^{(i)'}(n-1,q^n)\cap\pi_j=\{p_{ij}\}$, $i=1,2,\dots,q^{2n}$, $j=1,2,\dots,n$. Then $\pi_j=\{p_{1j},p_{2j},\dots,p_{q^{2n}j}\}\cup L_j$, with L_j the intersection of π_j with the extension of τ' to $\mathbf{GF}(q^n)$, $j=1,2,\dots,n$. Let M_j be a line of π_j, with $L_j\neq M_j$. Then $\langle M_j,\mathbf{PG}(n-1,q^n)\rangle$ intersects the extensions to $\mathbf{GF}(q^n)$ of all elements of a pseudo-oval $\mathcal{O}'$ on $\mathcal{O}$ containing $\mathbf{PG}(n-1,q)$. As $\mathcal{O}'$ is a regular pseudo-oval, there are planes γ_k over $\mathbf{GF}(q^n)$ intersecting the extensions to $\mathbf{GF}(q^n)$ of all elements of $\mathcal{O}'$, with $k=1,2,\ldots,n$. Then $\{\langle \mathbf{PG}(n-1,q^n),\gamma_k\rangle \cap \mathbf{PG}(3n-1,q^n) \parallel k=1,2,\ldots,n\} = \{\langle \mathcal{O}'\rangle \cap \pi_j \parallel j=1,2,\ldots,n\}$; here we rely on the fact that a regular $(n-1)$-spread in a $\mathbf{PG}(2n-1,q)$ uniquely defines n transversals over $\mathbf{GF}(q^n)$. So we may assume that $\langle \mathbf{PG}(n-1,q^n),\gamma_j\rangle \cap \mathbf{PG}(3n-1,q^n) = \langle \mathcal{O}'\rangle \cap \pi_j = M_j$, with $j=1,2,\ldots,n$. Clearly the extensions of the elements of $\mathcal{O}'$ intersect γ_j in the points of an oval $\mathcal{O}'_j$ over $\mathbf{GF}(q^n)$, with $\mathcal{O}'_j \cap \mathbf{PG}(n-1,q^n) = \{r_j\}$, $j=1,2,\ldots,n$. This oval $\mathcal{O}'_j$ is contained in $\langle \mathbf{PG}(n-1,q^n),\pi_j\rangle$. Now let N_j be the tangent to $\mathcal{O}'_j$ at r_j. Then $\langle N_j, \mathbf{PG}(n-1,q^n)\rangle$ intersects π_j in a point of L_j. If $\mathcal{O}'_j$ and $\mathcal{O}''_j$ are two such ovals, then we have the following two cases.

- If the corresponding lines of π_j intersect in a point not on L_j, then $(\mathcal{O}'_j \cap \mathcal{O}''_j) \setminus \mathbf{PG}(n-1,q^n)$ is a point.
- If the corresponding lines of π_j intersect in a point of L_j, then $(\mathcal{O}'_j \cap \mathcal{O}''_j) \setminus \mathbf{PG}(n-1,q^n) = \emptyset$ and they both define the same point of L_j.

Let the distinct points $p_{ij}, p_{i'j}, p_{i''j}$ define a triangle of π_j. The ovals defined by the lines $p_{ij}p_{i'j}, p_{i'j}p_{i''j}, p_{i''j}p_{ij}$ as above are denoted by $\mathcal{O}_j^{(2)}, \mathcal{O}_j^{(1)}, \mathcal{O}_j^{(3)}$ respectively. Then one easily sees that all $q^{2n}+q^n$ ovals $\mathcal{O}'_j$ are contained in the space $\langle \mathcal{O}_j^{(1)}, \mathcal{O}_j^{(2)}, \mathcal{O}_j^{(3)}\rangle$. So all these ovals generate a $\mathbf{PG}^{(j)}(m,q^n)$, with $3 \le m \le 5$.

(a) The Case $m=3$

Then $\langle \mathbf{PG}(n-1,q^n), \mathbf{PG}^{(j)}(3,q^n)\rangle = \langle \mathbf{PG}(n-1,q^n),\pi_j\rangle$ is $(n+2)$-dimensional, so $\mathbf{PG}(n-1,q^n) \cap \mathbf{PG}^{(j)}(3,q^n)$ is a point u. This point u belongs to the $q^{2n}+q^n$ ovals $\mathcal{O}'_j$. The union of these ovals is a set W of size $q^{2n}+1$. As no three points of W are collinear, this set W is an ovoid of $\mathbf{PG}^{(j)}(3,q^n)$. Hence the extensions of the elements of $\mathcal{O}$ meet the n spaces $\mathbf{PG}^{(1)}(3,q^n),\ldots,\mathbf{PG}^{(n)}(3,q^n)$, and the egg $\mathcal{O}$ is regular.

(b) The Case $m=5$

Then $\langle \mathbf{PG}(n-1,q^n), \mathbf{PG}^{(j)}(5,q^n)\rangle = \langle \mathbf{PG}(n-1,q^n),\pi_j\rangle$ is $(n+2)$-dimensional, so $\mathbf{PG}(n-1,q^n) \cap \mathbf{PG}^{(j)}(5,q^n)$ is a plane δ_j. Let $\mathbf{PG}^{(i)}(n-1,q^n) \cap \mathbf{PG}^{(j)}(5,q^n) = \{s_i\}$. Now we project from s_1s_2 onto a subspace $\mathbf{PG}(3,q^n)$

of $\mathbf{PG}^{(j)}(5,q^n)$ which is skew to s_1s_2. Let the q^n ovals containing s_1 but not s_2 be $\mathcal{O}_{11},\mathcal{O}_{12},\ldots,\mathcal{O}_{1q^n}$, and let the q^n ovals containing s_2 but not s_1 be $\mathcal{O}_{21},\mathcal{O}_{22},\ldots,\mathcal{O}_{2q^n}$. Further, let L_{li} be the tangent line to $\mathcal{O}_{li}$ at s_l, with $l=1,2$ and $i=1,2,\ldots,q^n$. By projection of $\mathcal{O}_{li}\setminus\{s_l\}$ from s_1s_2 onto $\mathbf{PG}(3,q^n)$, there arise q^n points of a line T_{li}. Let $\langle s_1,s_2,L_{li}\rangle\cap\mathbf{PG}(3,q^n)=t_{li}$; then $t_{li}\in T_{li}$. Assume, by way of contradiction, that $T_{li}\cap T_{lk}\neq\emptyset$ for $i\neq k$. Then all ovals in $\mathbf{PG}^{(j)}(5,q^n)$ are contained in $\langle T_{li},T_{lk},s_1,s_2\rangle$, which is at most 4-dimensional, a contradiction. So $T_{l1},T_{l2},\ldots,T_{lq^n}$ are mutually disjoint, $l=1,2$. If $\tilde{\mathcal{O}}$ is the oval containing s_1 and s_2, then $\tilde{\mathcal{O}}\setminus\{s_1,s_2\}$ is projected onto a point t of $\mathbf{PG}(3,q^n)$. Notations are chosen in such a way that $\mathcal{O}_{2i}$ is the oval on s_2 which contains no point of $\mathcal{O}_{1i}\setminus(\{s_1\}\cup\mathbf{PG}(n-1,q^n))$. So $T_{1i}\cap T_{2k}\neq\emptyset$ for $k\neq i$. It follows that there is a hyperbolic quadric $\mathcal{H}$ in $\mathbf{PG}(3,q^n)$ such that $\{T_{l1},\ldots,T_{lq^n}\}$ is a subset of a regulus $\mathcal{R}_l$ of $\mathcal{H}$. Let $\mathcal{O}_{li}\cap\mathbf{PG}(n-1,q^n)=\mathcal{O}_{li}\cap\delta_j=\{r_{li}\}$, and $\langle s_1,s_2,r_{li}\rangle\cap\mathbf{PG}(3,q^n)=\{s_{li}\}$. If $\overline{\tau_i}$ is the extension of τ_i to $\mathbf{GF}(q^n)$, then $\overline{\tau_i}\cap\mathbf{PG}^{(j)}(5,q^n)$ is a plane ρ_{ji}, with $j=1,2,\ldots,n$ and $i=1,2,\ldots,q^{2n}$. So the lines $L_{l1},L_{l2},\ldots,L_{lq^n}$ are contained in the plane ρ_{jl}. It follows that $t_{l1},t_{l2},\ldots,t_{lq^n}$ are points of some line T'_l, with $l=1,2$. Clearly $\{t\}=T'_1\cap T'_2$ and $s_{1i}=s_{2i}=s'_i$, with $i=1,2,\ldots,q^n$. Hence ovals which define a common point of L_j intersect δ_j in a common point. The points $s'_1,s'_2,\ldots,s'_{q^n}$ belong to $\langle\delta_j,s_1,s_2\rangle\cap\mathbf{PG}(3,q^n)$, so are contained in a plane ψ. Hence $s'_1,s'_2,\ldots,s'_{q^n}$ belong to the non-singular conic $\mathcal{C}=\psi\cap\mathcal{H}$. Also, we have $t\in\mathcal{C}$.

Now we consider an oval $\mathcal{O}'_j$ not containing s_1 nor s_2. By projection of $\mathcal{O}'_j$ from s_1s_2 there arises the intersection of $\mathcal{H}$ with a plane of $\mathbf{PG}(3,q^n)$ through t, but not containing T'_1 nor T'_2. Hence this intersection is a non-singular conic, and so $\mathcal{O}'_j$ is a conic. It follows that the corresponding pseudo-oval on $\mathcal{O}$ is a pseudo-conic. Now by M. R. Brown and M. Lavrauw [2] (see also J. A. Thas [28]) the egg is classical. But in such a case m is necessarily three, a contradiction.

(c) The Case $m=4$

Then $\langle\mathbf{PG}(n-1,q^n),\mathbf{PG}^{(j)}(4,q^n)\rangle=\langle\mathbf{PG}(n-1,q^n),\pi_j\rangle$ is $(n+2)$-dimensional, so $\mathbf{PG}(n-1,q^n)\cap\mathbf{PG}^{(j)}(4,q^n)$ is a line W_j. Let $\mathbf{PG}^{(i)}(n-1,q^n)\cap\mathbf{PG}^{(j)}(4,q^n)=\{s_i\}$, with $i=1,2,\ldots,q^{2n}$.

Let $w\in L_j$. With the q^n lines of π_j through w, but distinct from L_j, there correspond q^n ovals. The tangent lines to these ovals at their intersection point with W_j are contained in the plane $\mathbf{PG}^{(j)}(4,q^n)\cap\langle\mathbf{PG}(n-1,q^n),w\rangle$. Assume, by way of contradiction, that the point $u\in U=W_j$ is on at least

q^n+1 ovals $\mathcal{O}'_j$. So at least two of these ovals, say $\mathcal{O}_j^{(1)}$ and $\mathcal{O}_j^{(2)}$, have a point $s_i \notin U$ in common. Let $\mathcal{O}_j^{(3)}$ be a third oval containing u. As all ovals generate $\mathbf{PG}^{(j)}(4,q^n)$, the oval $\mathcal{O}_j^{(3)}$ cannot contain a point of both $\mathcal{O}_j^{(1)} \setminus \{u,s_i\}$ and $\mathcal{O}_j^{(2)} \setminus \{u,s_i\}$.

First, assume that $\mathcal{O}_j^{(3)}$ contains s_i. If $\overline{\tau_i}$ is the extension of τ_i to $\mathbf{GF}(q^n)$, then $\overline{\tau_i} \cap \mathbf{PG}^{(j)}(4,q^n)$ is a plane ρ_{ji}, with $j=1,2,\ldots,n$ and $i=1,2,\ldots,q^{2n}$. So the tangent lines at s_i to the ovals $\mathcal{O}_j^{(1)},\mathcal{O}_j^{(2)},\mathcal{O}_j^{(3)}$ are coplanar. It follows that these ovals are contained in the 3-dimensional space $\langle \rho_{ji},u\rangle$, and so all $q^{2n}+q^n$ ovals $\mathcal{O}'_j$ are contained in $\langle \rho_{ji},u\rangle$, clearly a contradiction.

So $\mathcal{O}_j^{(3)}$ (and any other oval $\mathcal{O}_j^{(k)}$ containing u, $k \neq 1,2$) does not contain s_i. Let $\mathcal{O}_j^{(k)}$ and $\mathcal{O}_j^{(k')}$ be distinct ovals containing u, with $|\mathcal{O}_j^{(k)} \cap \mathcal{O}_j^{(l)}| = |\mathcal{O}_j^{(k')} \cap \mathcal{O}_j^{(l)}| = 2$ and $l \in \{1,2\}$. Then $\mathcal{O}_j^{(k)} \cap \mathcal{O}_j^{(k')} = \{u\}$, as otherwise $\langle \mathcal{O}_j^{(l)},\mathcal{O}_j^{(k)},\mathcal{O}_j^{(k')}\rangle$ is 3-dimensional. That means that the ovals containing u are elements of two classes W_1 and W_2, where $\mathcal{O}_j^{(i)} \in W_i$ and any two distinct ovals of W_i just have u in common, with $i=1,2$. Now we project from u onto a $\mathbf{PG}(3,q^n)$ in $\mathbf{PG}^{(j)}(4,q^n)$, with $u \notin \mathbf{PG}(3,q^n)$. The ovals of W_i are denoted by $\mathcal{O}_{j1}^{(i)} = \mathcal{O}_j^{(i)}$, $\mathcal{O}_{j2}^{(i)},\ldots$, with $i=1,2$. Let $T_l^{(i)}$ be the tangent line to $\mathcal{O}_{jl}^{(i)}$ at u, and let $T_l^{(i)} \cap \mathbf{PG}(3,q^n) = \{t_l^{(i)}\}$. Then the points $t_1^{(i)},t_2^{(i)},\ldots$ are distinct, as otherwise all ovals are contained in a 3-dimensional space, with $i=1,2$; as ovals of different classes W_1 and W_2 always have a point of $\mathbf{PG}^{(j)}(4,q^n) \setminus U$ in common, we clearly have $t_k^{(1)} \neq t_{k'}^{(2)}$. Let the projection from u onto $\mathbf{PG}(3,q^n)$ of $\mathcal{O}_{jl}^{(i)} \setminus \{u\}$ belong to the line $V_l^{(i)}$. Then $t_l^{(i)} \in V_l^{(i)}$. Now we shall prove that the lines $V_l^{(i)}$, $i=1,2$, all belong to a common hyperbolic quadric $\mathcal{H}$ of $\mathbf{PG}(3,q^n)$. Suppose, by way of contradiction, that these lines do not belong to a common hyperbolic quadric. As $T_1^{(i)},T_2^{(i)},\ldots$ define a common point on L_j, the points $t_1^{(i)},t_2^{(i)},\ldots$ are collinear, with $i=1,2$. Assume that $|W_1| \leq |W_2|$. The only possibility now is that $|W_1|=1$ and $|W_2|=q^n$. The points $t_1^{(2)},t_2^{(2)},\ldots$ are on a line R; let u' be the (q^n+1)-th point of that line. Then $\{u'\}=U \cap \mathbf{PG}(3,q^n)$. With the $q^{2n}-1$ ovals not containing u correspond, by projection from u, $q^{2n}-1$ ovals through u'. Any such projection is a subset of $V_1^{(2)} \cup V_2^{(2)} \cup \cdots \cup V_{q^n}^{(2)} \cup \{u'\}$. The planes of these projections, together with the q^n+1 planes on R and the plane $\langle u',V_1^{(1)}\rangle$, are the $q^{2n}+q^n+1$ planes of $\mathbf{PG}(3,q^n)$ through u'. Now consider a line V, with $V_1^{(1)} \neq V \neq R$, which intersects $V_1^{(2)}, V_2^{(2)}, V_3^{(2)}$. Then $\langle V, u'\rangle$ intersects $V_1^{(2)} \cup V_2^{(2)} \cup \cdots \cup V_{q^n}^{(2)} \cup \{u'\}$ in q^n+1 points which form an oval, a contradiction as this intersection contains three distinct collinear points. Consequently the lines $V_l^{(i)}$ all belong to a common hyperbolic quadric $\mathcal{H}$ of $\mathbf{PG}(3,q^n)$. Assume $|W_1| \leq |W_2|$ and let $U \cap \mathbf{PG}(3,q^n) = \{u'\}$; from the foregoing it follows that $|W_1| \geq 2$. Then the points $u', t_1^{(i)}, t_2^{(i)}, \ldots$ are on a common line R_i, $i = 1,2$. Now we consider an oval $\mathcal{O}'_j$ with $\mathcal{O}_j^{(1)} \cap \mathcal{O}'_j = \emptyset$ (such an oval exists if we are not in the case $|W_1| = |W_2| = q^n$). The projection $\overline{\mathcal{O}_j}'$ of $\mathcal{O}'_j$ from u contains u', and the projection of the tangent line to $\mathcal{O}'_j$ at $\mathcal{O}'_j \cap U$ is the line R_1. It follows that $\langle \overline{\mathcal{O}_j}' \rangle$ contains R_1. As $R_1 \subseteq \mathcal{H}$ the oval $\overline{\mathcal{O}'_j}$ contains $|W_2| \geq q^n - 1$ points of a line of $\mathcal{H}$, clearly a contradiction. Hence $|W_1| = |W_2| = q^n$. Any oval $\mathcal{O}'_j$ not in $W_1 \cup W_2$ is then projected onto a conic of $\mathcal{H}$, and so $\mathcal{O}'_j$ is a conic. It follows that the corresponding pseudo-oval on $\mathcal{O}$ is a pseudo-conic. Now by M. R. Brown and M. Lavrauw [2] (see also J. A. Thas [28]) the egg $\mathcal{O}$ is classical. But in such a case m is necessarily three, a contradiction.

So we have shown that each point u of U is contained in exactly q^n of the $q^{2n} + q^n$ ovals.

Consider again two ovals $\mathcal{O}_j^{(1)}, \mathcal{O}_j^{(2)}$ containing the point $u \in U$ and assume that $\mathcal{O}_j^{(1)} \cap \mathcal{O}_j^{(2)} = \{u, s_i\}$, with $s_i \notin U$. Then as before all ovals containing u are elements of two classes W_1 and W_2, where $\mathcal{O}_j^{(i)} \in W_i$ and any two distinct ovals of W_i just have u in common, with $i = 1,2$. Assume that $|W_1| \leq |W_2|$. We proceed as before, using similar notation. Then, if the lines $V_l^{(i)}$ do not belong to a common hyperbolic quadric $\mathcal{H}$, we have $|W_1| = 1$ and $|W_2| = q^n - 1$. In such a case the planes of the projections of the q^{2n} ovals not on u, together with the $q^n + 1$ planes through R in $\mathbf{PG}(3,q^n)$ and the plane $\langle u', V_1^{(1)}\rangle$, are the $q^{2n} + q^n + 1$ distinct planes in $\mathbf{PG}(3,q^n)$ containing u'; the unique oval $\mathcal{O}^*$ having no point in common with the union of the elements of W_2 defines a plane of $\mathbf{PG}(3,q^n)$ which contains R (the projection $\tilde{\mathcal{O}}^*$ onto $\mathbf{PG}(3,q^n)$ of the oval is tangent to R at u'). Now consider a line V, with $V_1^{(1)} \neq V \neq R$, which intersects $V_1^{(2)}, V_2^{(2)}, V_3^{(2)}$. Then $\langle V, u'\rangle$ intersects $V_1^{(2)} \cup V_2^{(2)} \cup \cdots \cup V_{q^n-1}^{(2)} \cup \tilde{\mathcal{O}}^* \cup \{u'\}$ in $q^n + 1$ points which form an oval, a contradiction as this intersection contains three distinct collinear points. So the lines $V_l^{(i)}$ belong to a common hyperbolic quadric $\mathcal{H}$ of $\mathbf{PG}(3,q^n)$. Also, from the foregoing it follows that $|W_1| \geq 2$. Consider an oval $\mathcal{O}'_j$ with $\mathcal{O}_j^{(1)} \cap \mathcal{O}'_j = \emptyset$. The projection $\tilde{\mathcal{O}}'_j$ of $\mathcal{O}'_j$ from u onto $\mathbf{PG}(3,q^n)$ contains u', and the projection of the tangent line to $\mathcal{O}'_j$ at $\mathcal{O}'_j \cap U$ is the line R_1 containing

$u', t_1^{(1)}, t_2^{(1)}, \ldots$. It follows that $\langle \tilde{\mathcal{O}}'_j \rangle$ contains R_1. As R_1 contains at least three points of $\mathcal{H}$, it belongs to $\mathcal{H}$. It follows that $\tilde{\mathcal{O}}'_j$ contains $|W_2| \geq q^n - 2$ points of a line of $\mathcal{H}$, clearly a contradiction.

So if $\mathcal{O}_j^{(1)}, \mathcal{O}_j^{(2)}$ are any two distinct ovals containing a point u of U, then $\mathcal{O}_j^{(1)} \cap \mathcal{O}_j^{(2)} = \{u\}$. Let $u \in U$ and project from u onto a $\mathbf{PG}(3, q^n) \subseteq \mathbf{PG}^{(j)}(4, q^n)$, with $u \notin \mathbf{PG}(3, q^n)$. Let $\mathcal{O}_j^{(1)}, \mathcal{O}_j^{(2)}, \ldots, \mathcal{O}_j^{(q^n)}$ be the ovals containing u, and let T_i be the tangent line to $\mathcal{O}_j^{(i)}$ at u. Further, let $T_i \cap \mathbf{PG}(3, q^n) = \{t_i\}$ and let the projection of $\mathcal{O}_j^{(i)} \setminus \{u\}$ belong to the line V_i, with $i = 1, 2, \ldots, q^n$. Then $t_i \in V_i$. The points $t_1, t_2, \ldots, t_{q^n}$ are not necessarily distinct. Also, as $T_1, T_2, \ldots, T_{q^n}$ define a common point on L_j, the points $t_1, t_2, \ldots, t_{q^n}$ are collinear. If $U \cap \mathbf{PG}(3, q^n) = \{u'\}$, then the points $u', t_1, t_2, \ldots, t_{q^n}$ are on a common line U' of $\mathbf{PG}(3, q^n)$. Let $u_l \in U \setminus \{u\}$, $l = 1, 2, \ldots, q^n$. The tangent lines at u_l to the q^n ovals on u_l are coplanar, so intersect $\mathbf{PG}(3, q^n)$ in points on a line U'_l containing u'. In this way there arise q^n lines $U'_1, U'_2, \ldots, U'_{q^n}$, which, together with U', are the $q^n + 1$ lines through u' in a plane π'. The projections of the q^{2n} ovals not through u are projected onto ovals in the q^{2n} planes of $\mathbf{PG}(3, q^n)$ which contain the point u' but not the line U'. Each of these ovals in $\mathbf{PG}(3, q^n)$ contains u' and a point of $V_i \setminus U'$, with $i = 1, 2, \ldots, q^n$. If $t_i = t_k$, $i \neq k$, then $\langle \mathcal{O}_j^{(i)}, \mathcal{O}_j^{(k)} \rangle$ does not contain U, as otherwise all ovals would be contained in a 3-dimensional space. If $t_i \neq t_k$, $i \neq k$, then we can find three collinear points on $(V_1 \cup V_2 \cup \cdots \cup V_{q^n}) \setminus U'$ such that the line W' joining them has no point in common with U' (if e.g. $t_i \in V_i$, $t_i = t_l \in V_l$, $t_k \in V_k$, with i, l, k distinct, then in the plane $\langle V_i, V_l \rangle$ there are three collinear points on $V_i \cup V_l \cup V_k$ satisfying the requirement). Then $\langle W', u' \rangle$ contains the projection $\overline{\mathcal{O}_j}'$ of an oval not containing u, and $\overline{\mathcal{O}_j}'$ has three collinear points, a contradiction. Consequently $t_1 = t_2 = \cdots = t_{q^n}$. Hence $T_1 = T_2 = \cdots = T_{q^n}$. A similar conclusion holds for each point $u_l \in U$, $l = 1, 2, \ldots, q^n$. So in $\mathbf{PG}^{(j)}(4, q^n)$ the nuclei of the $q^{2n} + q^n$ ovals are on $q^n + 1$ lines $B, B_1, \ldots, B_{q^n}$. As $B, B_1, \ldots, B_{q^n}$ define all $q^n + 1$ points of L_j, these lines are mutually skew. In the tangent space τ to $\mathcal{O}$ at $\mathbf{PG}(n-1, q)$, the space $\mathbf{PG}(n-1, q)$ together with the nuclei of the pseudo-ovals on $\mathcal{O}$ containing $\mathbf{PG}(n-1, q)$ form a regular $(n-1)$-spread S in τ; see Section 4 of J. A. Thas [24]. As the nuclei of the $q^{2n} + q^n$ pseudo-ovals are distinct, also the nuclei of the $q^{2n} + q^n$ ovals are distinct. The extensions to $\mathbf{GF}(q^n)$ of the elements of S meet n planes $\gamma_1, \gamma_2, \ldots, \gamma_n$ over $\mathbf{GF}(q^n)$. Hence $B, B_1, \ldots, B_{q^n}$ are contained in distinct planes γ_i, a contradiction as $q^n + 1 > n$. Now the theorem is completely proved. ∎

10. Geometric Proof of a Theorem of Johnson

Let $\mathcal{F}$ be a *flock* of a quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3,q)$, i.e., a partition of the cone minus its vertex into q disjoint irreducible conics. Then by work of W. M. Kantor [8], S. E. Payne [12] and J. A. Thas [23], there can be constructed a GQ of order (q^2,q) from $\mathcal{F}$, denoted $\mathcal{S}(\mathcal{F})$ and called 'flock generalized quadrangle'.

If the TGQ $T(\mathcal{O})$ of order (s,s^2) is the dual of a flock GQ, then, by S. E. Payne [14] and N. L. Johnson [7], the flock is a (derived flock of a) semifield flock; further, for s even, Johnson [7] proves that such a flock is linear and so $T(\mathcal{O})$ is classical. As a corollary of the previous theorem we will give a new purely geometric proof of Johnson's theorem. Another short proof of that theorem is contained in L. Storme and J. A. Thas [19], using essentially a coordinatization method.

Theorem 10.1 (N. L. Johnson [7]). *If the TGQ $T(\mathcal{O})$ of order (s,s^2), with s even, is the dual of a flock GQ, then $T(\mathcal{O})$ is classical.*

Proof. Let $T(\mathcal{O})$ be a TGQ of order (s,s^2), s even, which is the dual of a flock GQ. By Section 4 of J. A. Thas [24] the egg $\mathcal{O}$ is good at some element $\mathbf{PG}(n-1,q)$, where $s=q^n$. Let $\mathcal{O}'$ be any of the pseudo-ovals on $\mathcal{O}$ through $\mathbf{PG}(n-1,q)$. By S. E. Payne [13] the TGQ $T(\mathcal{O}')^D$ of order q^n is isomorphic to a $T_2(\tilde{\mathcal{O}}')$ of Tits, with $\tilde{\mathcal{O}}'$ some oval in $\mathbf{PG}(2,q^n)$; the base-point of $T(\mathcal{O}')^D$ is the line $\mathbf{PG}(n-1,q)$ of $T(\mathcal{O}')$. As the elements of $\mathcal{O}'$ are regular points of $T(\mathcal{O}')^D$, the GQ $T(\mathcal{O}')$ is a $T_2(\mathcal{O}'')$ of Tits, where $\mathcal{O}''$ is a translation oval of $\mathbf{PG}(2,s)$; see Chapter 12 of FGQ. It follows that the kernel of $T(\mathcal{O}')$ is $\mathbf{GF}(q^n)$, and so $\mathcal{O}'$ is regular. Now by the preceding theorem the egg $\mathcal{O}$ is regular, that is, $T(\mathcal{O})$ is isomorphic to a $T_3(\tilde{\mathcal{O}})$ of Tits, with $\tilde{\mathcal{O}}$ some ovoid of $\mathbf{PG}(3,q^n)$. Let x be the point of $\tilde{\mathcal{O}}$ which corresponds to $\mathbf{PG}(n-1,q)\in\mathcal{O}$. By the foregoing all ovals on $\tilde{\mathcal{O}}$ through x are translation ovals. By C. E. Praeger and T. Penttila [16], $\tilde{\mathcal{O}}$ is either an elliptic quadric or a Tits ovoid. As for a non-classical flock GQ the special point (∞) is unique, and the automorphism group of the Tits ovoid acts transitively on the ovoid, this case cannot occur. Thus $\tilde{\mathcal{O}}$ is an elliptic quadric, and $T_3(\tilde{\mathcal{O}})\cong T(\mathcal{O})$ is classical. ∎

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